An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 40 (2000) 1405–1418
www.elsevier.nl/locate/camwa

Finite-Dimensional Dynamics of Distributed Optical Systems with Delayed Feedback

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Abstract—A mathematical approach for evaluation the complexity of spatio-temporal dynamics for delayed feedback optical systems is suggested. It is based on the concept of global attractor for discrete semigroups. Both the retarded differential-difference and retarded diffusion models of feedback optical systems are similarly investigated. Upper estimates for the number of determining modes and Hausdorff dimension of attractors are obtained.

Keywords—Nonlinear optical systems, Delay, Retarded differential equations, Attractor, Hausdorff dimension.

1. INTRODUCTION

Recently, nonlinear optical systems with a wide range of spatio-temporal field transformations in the distributed feedback loop were intensively studied as promising tools in optical information processing [1,2]. Rich opportunities in the creation of various types of local and nonlocal feedback interactions permit us to construct a variety of self-organization processes [2,3]: running pulses, dissipative structures, leading centers, reverberators, spirals, optical turbulence, etc. In principle, such spatio-temporal patterns can be used for coding and storing information (for instance, memory functions in [4]). The latest achievements in the developing of controlled delay line [5] turn one to the practical stage of pattern recognition via distributed optical systems. In [5], some experiments on the learning of temporal information are presented in a delayed optical system (resonator); a self-organized association of analog signals with resonator modes is demonstrated as well. The analog method for searching a global extremum for a function of many variables is suggested in [6] on the basis of specially organized feedback field interactions. As a matter of fact, the practical realization of such an optical system appears as an example of a special "optical computer" capable of carrying out a global optimization procedure.

Problems of quantitative estimating of the variety of all possible dynamic patterns and measuring of their complexity arise while using the distributed optical system in information processing.

The research described in this publication was made possible in part by Grant Nos. MRP000 and MRP300 from the International Science Foundation.

0898-1221/00/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0898-1221(00)00249-2

These are closely connected with a studying of the capacity for spatial chaos memory [4]. Roughly speaking, it is important to estimate the information store the optical system could accumulate by itself. A global attractor (if it exists) is known as a appropriate mathematical model: it contains full information of all limiting (as time tends to infinity) regimes possessed by the system. Thus, we come to the problem of estimating the attractor dimension.

The existence of global attractor is proved in [7,8] for the nonlinear optical system with nonlocal interactions provided by field rotation in the spatially distributed feedback contour without delay. A corresponding mathematical model is presented by the functional-differential diffusion equation with transformed spatial argument in the nonlinear term. The lower and upper estimates of Hausdorff dimension of the global attractor are obtained as well as the number of determining modes.

Taking account of the temporal delay in the feedback loop causes new problems. A phase space of the delayed systems becomes infinite dimensional. Even in a relatively simple model, described by a retarded differential-difference equation, the transition to various chaotic regimes is a quite natural process [4,9]. Lyapunov exponents are used in [4,10] when studying the system's complexity; it is mentioned that Lyapunov dimension of the chaotic attractor linearly increases with the delay value. Nonlinear optical systems having both temporal (delay in time) and spatial (diffusion in nonlinear media and nonlocal feedback transformations) field interactions form a new class of models. Their investigation has just begun.

In the present paper, we intend the mathematical approach for evaluation of the complexity of spatio-temporal dynamics for delayed feedback optical systems. Basing on the concept of global attractor for discrete semigroups, we demonstrate how both the retarded differential-difference and retarded diffusion models of feedback optical systems can be similarly investigated. The paper is organized as follows. In Section 2, we consider a family of time-delayed models including a chain of coupled optical elements and systems with spatially distributed feedback and field rotation. Corresponding mathematical models are described by retarded differential-difference and retarded diffusion equations with a transformed spatial variable. The reduction technique is discussed in Section 3; it is to be used as background for the treatment of retarded systems by means of discrete semigroups. We prove that they possess compact global attractors. In Section 4, we develop a method for finding quantitative estimates of "finite dimensionality" of attractors in retarded differential-difference equations and retarded diffusion equations with a transformed spatial variable. As a result, we obtain upper estimates of the number of determining modes and of the Hausdorff dimension of global attractors. In the conclusion, we put final comments; the Appendix contains the derivation of some formulas.

2. BASIC MODELS OF TIME-DELAYED OPTICAL SYSTEMS

The simplest (in form but not in the behavior) model of delayed optical system is considered in [9]. Using geometric-optical approximation of light beams' interactions in the one-dimensional feedback contour, authors have obtained the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(\mathbf{r},t) + u(\mathbf{r},t) = K\left(1 + \gamma\cos\{u(\mathbf{r},t-T)\}\right). \tag{1}$$

Here, $u(\mathbf{r},t)$ is a nonlinear phase modulation occurring in the thin ring aperture of radius r_0, $\mathbf{r} = (x,y) \in R^2$, $|\mathbf{r}|^2 \equiv x^2+y^2 = r_0^2$, time variable is normalized by its relaxation value. Note that in equation (1), the dependence on the spatial variable $\mathbf{r}$ is parametric, and therefore, different points of nonlinear medium are not connected with each other.

Introduction of the field rotation $\Delta_n = 2\pi/n$ (n is an integer number) in a feedback contour results in the appearance of interactions between each family of n points, equally spaced on the thin ring aperture. The dynamics of the system is described by vector-valued n-dimensional phase modulation $\mathbf{u}(t) = (u_1(t), u_2(t), \dots, u_n(t))^{\mathsf{T}} \in R^n$ governed by the system of retarded differential-difference equations [11]

$$\begin{aligned}
\frac{\mathrm{d}u_1}{\mathrm{d}t}(t) + u_1(t) &= K\left(1+\gamma\cos\left\{u_2(t-T)\right\}\right),\\
&\vdots\\
\frac{\mathrm{d}u_j}{\mathrm{d}t}(t) + u_j(t) &= K\left(1+\gamma\cos\left\{u_{j+1}(t-T)\right\}\right),\\
&\vdots\\
\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) + u_n(t) &= K\left(1+\gamma\cos\left\{u_1(t-T)\right\}\right).
\end{aligned} \tag{2}$$

The same equations also describe nonlinear field dynamics in a chain of n coupled optical elements [4].

We are going to generalize the models (1),(2) to the following system given in vector form:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) + \mathbf{u}(t) = \mathbf{F}\left(\mathbf{u}(t-T)\right), \tag{3}$$

where a vector-valued function

$$\mathbf{F}(\mathbf{z}) = \left(f_1(\mathbf{z}), f_2(\mathbf{z}), \dots, f_n(\mathbf{z})\right)^{\top}, \qquad \mathbf{z} = (z_1, z_2, \dots, z_n) \in R^n,$$

is specified by the type of nonlinear medium and the form of light beams' interactions in the feedback contour. We suppose that the following conditions of its growth are satisfied:

$$\max_{j=1,2,\dots,n} |f_j(\mathbf{z})| \le M_0, \qquad \max_{j,l=1,2,\dots,n} \left|\frac{\partial f_j}{\partial z_l}(\mathbf{z})\right| \le M_1. \tag{4}$$

Note that the right part of (2) satisfies (4) with constants $M_0 = K(1+\gamma)$, $M_1 = K\gamma$.

The unique solution of (3) may be obtained after introduction of the initial conditions

$$\begin{aligned}
\mathbf{u}(t-T) &= \mathbf{\Phi}^{(-1)}(t), \qquad 0 \le t < T,\\
\mathbf{u}(0) &= \mathbf{\Phi}^{(0)}.
\end{aligned} \tag{5}$$

The next step in developing of the mathematical model is based on taking into account nonlocal field interactions provided by both spatial rotation and temporal delay in feedback contour together with local interactions of diffusion type. This approach is suggested in [3,11], the corresponding retarded diffusion type equation with transformed spatial variable appears as follows:

$$\frac{\partial u}{\partial t} + u = D\frac{\partial^2 u}{\partial x^2} + K\left(1+\gamma\cos\left\{u(x+\Delta, t-T)\right\}\right). \tag{6}$$

Here, $D = dr_0^{-2}$, d is a diffusion coefficient, Δ is a rotation angle in a feedback contour. Unlike the model (2) in (6) phase modulation $u = u(x,t)$ 2π-periodically depends on the continuously varied angle variable x.

The general model of distributed optical system with above-mentioned local and nonlocal field interactions may be represented by the equation

$$\frac{\partial u}{\partial t} + u = D\frac{\partial^2 u}{\partial x^2} + F\left(u(x+\Delta, t-T)\right) \tag{7}$$

of 2π-periodic in angle phase modulation $u(x,t)$ and the following initial conditions:

$$u(x, t-T) = \varphi^{(-1)}(x,t), \qquad 0 \le t < T, \quad x \in [0, 2\pi], \tag{8}$$

$$u(x, 0) = \varphi^{(0)}(x), \qquad x \in [0, 2\pi]. \tag{9}$$

A scalar-valued function F, which describes a particular type of nonlinear interaction in the feedback contour, is supposed to satisfy the conditions

$$|F(z)| \le M_0, \qquad |F'(z)| \le M_1. \tag{10}$$

In the next sections, we will show that the dynamics of systems (3)–(5) and (7)–(10) may be similarly investigated by the reduction to discrete semigroups.

3. THE REDUCTION OF THE DELAYED SYSTEMS AND THE EXISTENCE OF GLOBAL ATTRACTORS

In the theory of retarded differential equations, the operator of shifting along the trajectory over the delay period is used to represent the solutions by means of a discrete dynamical system [12]. Basing on this idea, we introduce the operator S according to the formula

$$S : \left(\mathbf{\Phi}^{(-1)}, \mathbf{\Phi}^{(0)}\right) \to (\mathbf{u}, \mathbf{u}(T)),$$

where $\mathbf{u}(T) = \mathbf{u}(t)|_{t=T}$, $\mathbf{u}(t)$ is the solution of the classical Cauchy problem for the system of ordinary differential equations

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) + \mathbf{u}(t) = \mathbf{F}\left(\mathbf{\Phi}^{(-1)}(t)\right), \qquad 0 < t < T, \tag{11}$$

$$\mathbf{u}(0) = \mathbf{\Phi}^{(0)}. \tag{12}$$

Operator S acts in the Hilbert space of pairs $(\mathbf{\Phi}^{(-1)}, \mathbf{\Phi}^{(0)}) \in \mathcal{H} = L_2^n(0,T) \times R^n$, where Euclidean n-dimensional space R^n is provided by the scalar product and corresponding norm

$$\left\langle \mathbf{\Phi}^{(0)}, \mathbf{\Psi}^{(0)} \right\rangle_n = \sum_{j=1}^{n} \mathbf{\Phi}_j^{(0)} \mathbf{\Psi}_j^{(0)}, \qquad \left\| \mathbf{\Phi}^{(0)} \right\|_n^2 = \left\langle \mathbf{\Phi}^{(0)}, \mathbf{\Phi}^{(0)} \right\rangle_n.$$

The scalar product and the norm in the Lebesgue space $L_2^n(0,T)$ are

$$\left\langle \mathbf{\Phi}^{(-1)}, \mathbf{\Psi}^{(-1)} \right\rangle_{(0,T),n} = \int_0^T \left\langle \mathbf{\Phi}^{(-1)}(t), \mathbf{\Psi}^{(-1)}(t) \right\rangle_n \mathrm{d}t,$$

$$\left\| \mathbf{\Phi}^{(-1)} \right\|_{(0,T),n}^2 = \left\langle \mathbf{\Phi}^{(-1)}, \mathbf{\Phi}^{(-1)} \right\rangle_{(0,T),n}.$$

Then, for arbitrary given elements $\mathbf{\Phi} = (\mathbf{\Phi}^{(-1)}, \mathbf{\Phi}^{(0)}) \in \mathcal{H}$ and $\mathbf{\Psi} = (\mathbf{\Psi}^{(-1)}, \mathbf{\Psi}^{(0)}) \in \mathcal{H}$ their scalar product and the norm are determined as follows:

$$\langle \mathbf{\Phi}, \mathbf{\Psi} \rangle_{\mathcal{H}} = \left\langle \mathbf{\Phi}^{(-1)}, \mathbf{\Psi}^{(-1)} \right\rangle_{(0,T),n} + \left\langle \mathbf{\Phi}^{(0)}, \mathbf{\Psi}^{(0)} \right\rangle_n, \qquad \|\mathbf{\Phi}\|_{\mathcal{H}}^2 = \langle \mathbf{\Phi}, \mathbf{\Phi} \rangle_{\mathcal{H}}.$$

Using these definitions, we reduce the dynamics of (3)–(5) to the iterations of operator S

$$\mathbf{u}(t; \mathbf{\Phi}) = S^k \mathbf{\Phi}\big|_{(-1)} (t - (k-1)T), \qquad (k-1)T < t < kT, \quad k = 1, 2, \dots. \tag{13}$$

Here and below, for every $\mathbf{\Phi} = (\mathbf{\Phi}^{(-1)}, \mathbf{\Phi}^{(0)}) \in \mathcal{H}$ we denote $\mathbf{\Phi}|_{(-1)} \equiv \mathbf{\Phi}^{(-1)}$, $\mathbf{\Phi}|_{(0)} \equiv \mathbf{\Phi}^{(0)}$; $\mathbf{u}(t; \mathbf{\Phi})$ denotes a solution of (3)–(5) started from the initial data $\mathbf{\Phi}$. Thus, the solution's behavior as $t \to +\infty$ may be investigated by examining of the properties of iteration operator S and its powers.

Observing the method of reduction, one can see that it remains correct for the retarded system (7)–(10) with spatially distributed parameters. We introduce the iteration operator S_D according to the rule

$$S_D : \left(\varphi^{(-1)}, \varphi^{(0)}\right) \to (u, u(T)),$$

where $u(T) = u(x,t)|_{t=T}$, $u(x,t)$ is a 2π-periodic with respect to x solution of the initial-boundary value problem for the linear diffusion equation

$$\frac{\partial u}{\partial t} + u = D\frac{\partial^2 u}{\partial x^2} + F\left(\varphi^{(-1)}(x + \Delta, t)\right), \qquad 0 < t \le T, \tag{14}$$

$$u(x, 0) = \varphi^{(0)}(x). \tag{15}$$

Operator S_D acts in the space $\mathcal{H}_D = L^2(\Omega_T) \times L^2(\Omega)$, $\Omega_T = \Omega \times (0,T)$, $\Omega = (0, 2\pi)$. Here we use Lebesgue spaces $L^2(\Omega)$ and $L^2(\Omega_T)$ of square integrable over Ω and Ω_T functions with standard scalar products and corresponding norms further referred as $\langle\cdot,\cdot\rangle_\Omega$, $\|\cdot\|_\Omega$ and $\langle\cdot,\cdot\rangle_{\Omega_T}$, $\|\cdot\|_{\Omega_T}$. Provided by the scalar product

$$\langle \Phi, \Psi\rangle_{\mathcal{H}_D} = \left\langle \varphi^{(-1)}, \psi^{(-1)} \right\rangle_{\Omega_T} + \left\langle \varphi^{(0)}, \psi^{(0)} \right\rangle_{\Omega},$$

of elements $\Phi = (\varphi^{(-1)}, \varphi^{(0)})$, $\Psi = (\psi^{(-1)}, \psi^{(0)})$ and corresponding norm $\|\cdot\|_{\mathcal{H}_D}$ the space $\mathcal{H}_D$ becomes a Hilbert one.

The spatially-distributed analogue of formula (13) may be written as follows:

$$u(t;\Phi) = S_D^k\Phi\big|_{(-1)}(t-(k-1)T), \qquad (k-1)T < t < kT, \quad k = 1,2,\dots. \tag{16}$$

According to (13) and (16) the iteration operators S and S_D play the same role in the construction of the solutions and prove to be a convenient instrument for examining the dynamics of delayed systems. Really, as it is followed from (13) and (16), trajectories of delayed systems may be represented by the powers of iteration operators

$$\mathbf{\Phi}_k = S^k\mathbf{\Phi}, \quad \mathbf{\Phi} \in \mathcal{H}; \qquad \Phi_k = S_D^k\Phi, \quad \Phi \in \mathcal{H}_D, \quad k = 1,2,\dots.$$

Thus, we may use functional consequences $\{\mathbf{\Phi}_k\}_{k=0}^{+\infty}$ and $\{\Phi_k\}_{k=0}^{+\infty}$ with new discrete "temporal" variable "k" instead of functions $\mathbf{u}(t)$ and $u(x,t)$ dependent on continuously valued variable t.

Facing to the investigation of discrete dynamical systems $\{S^k\mathbf{\Phi}, \mathbf{\Phi} \in \mathcal{H}\}$ and $\{S_D^k\Phi, \Phi \in \mathcal{H}_D\}$ we point out that they are represented in the form of semigroups, acting in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_D$. To examine their properties, let us consider the inequalities obtained in the Appendix:

$$\left\|S^k\mathbf{\Phi}\right\|_{\mathcal{H}}^2 \le \|\mathbf{\Phi}\|_{\mathcal{H}}^2 \exp(-(k-1)T) + nM_0^2(1+T), \qquad \mathbf{\Phi} \in \mathcal{H}, \tag{17}$$

$$\left\|S_D^k\Phi\right\|_{\mathcal{H}_D}^2 \le \|\Phi\|_{\mathcal{H}_D}^2 \exp(-(k-1)T) + 2\pi M_0^2(1+T), \qquad \Phi \in \mathcal{H}_D. \tag{18}$$

From (17) we deduce that all solutions $\mathbf{\Phi}_k = S^k\mathbf{\Phi}$, started from the ball $\{\mathbf{\Phi} \in \mathcal{H} : \|\mathbf{\Phi}\|_{\mathcal{H}} \le \rho\}$, remain in a bounded set for all $k = 1,2,\dots$

$$\|\mathbf{\Phi}_k\|_{\mathcal{H}}^2 \le \rho^2 + nM_0^2(1+T).$$

It means that semigroup $\{S^k\}$ is uniformly bounded. More than that, there exists an absorbing set of the semigroup. For example, the ball $B = \{\mathbf{\Phi} \in \mathcal{H} : \|\mathbf{\Phi}\|_{\mathcal{H}}^2 \le 1 + nM_0^2(1+T)\}$ satisfies the condition of absorbing: for every $\mathbf{\Phi} \in \mathcal{H}$, there exists the moment of time $k_0 = k_0(\mathbf{\Phi})$, that $S^k\mathbf{\Phi} \subset B$ for every $k \ge k_0$. Here we may get $k_0(\mathbf{\Phi}) = 2 + [2T^{-1}\ln\|\mathbf{\Phi}\|_{\mathcal{H}}]$, where $[\cdot]$ is the notation of integer part. Similar conclusions are valid for semigroup $\{S_D^k\}$ (here we use (18)). Thus, the dynamics of considered delayed optical systems may be characterized as dissipative.

Another feature of the semigroups is connected with the compactness of operators S and S_D. Really, for the systems of delayed differential equations, the compactness property is a well-known fact (see [12]). It follows from the existence of the derivative $\frac{\mathrm{du}}{\mathrm{dt}}$ bounded in $L_2^n(0,T)$. In the case of operator $S_D : \mathcal{H}_D \to \mathcal{H}_D$, we may refer to the properties of solutions for diffusion type equations (see for instance [13]), specified here in terms of Sobolev spaces with fractional derivatives

$$S_D\Phi|_{(-1)} \equiv u(x,t;\Phi) \in H^{1,1/2}(\Omega_T), \qquad S_D\Phi|_{(0)} \equiv u(x,T;\Phi) \in H^1(\Omega).$$

Then the compactness of S_D follows from the compactness of inclusions [13]

$$H^{1,1/2}(\Omega_T) \times H^1(\Omega) \hookrightarrow L^2(\Omega_T) \times L^2(\Omega) \equiv \mathcal{H}_D.$$

The dissipativeness and compactness of the semigroups provide the existence of nonempty sets

$$\mathcal{U} = \bigcap_{s=1}^{+\infty} \mathrm{cl}_{\mathcal{H}} \left\{ \bigcup_{k=s}^{+\infty} S^k(B) \right\}, \qquad \mathcal{U}_D = \bigcap_{s=1}^{+\infty} \mathrm{cl}_{\mathcal{H}_D} \left\{ \bigcup_{k=s}^{+\infty} S_D^k(B_D) \right\},$$

where $\mathrm{cl}\{\cdot\}$ denotes the closure in corresponding spaces; B and B_D are absorbing sets of semigroups. It is important that $\mathcal{U}$ and $\mathcal{U}_D$ are independent on particular absorbing set and are global attractors of semigroups $\{S^k\}$ and $\{S_D^k\}$, correspondingly. These facts follow from general results obtained for dissipative compact semigroups (see [12]).

Let us consider the notion of global attractor, taking for definiteness semigroup $\{S^k\}$. The global attractor $\mathcal{U}$ of semigroup $\{S^k\}$ appears as a compact set in $\mathcal{H}$ satisfying the following properties (see among others [12,14–16]).

(i) The *invariance property*: $S(\mathcal{U}) = \mathcal{U}$, i.e., if $\mathbf{\Phi} \in \mathcal{U}$, then $S\mathbf{\Phi} \in \mathcal{U}$; and vice versa, for all $\mathbf{\Psi} \in \mathcal{U}$, there exists $\mathbf{\Phi} \in \mathcal{U}$ that $\mathbf{\Psi} = S\mathbf{\Phi}$.

(ii) The *attracting property*:

$$\lim_{k\to+\infty} \mathrm{dist}_{\mathcal{H}} \left(S^k(B), \mathcal{U}\right) = 0$$

for every bounded set $B \subset \mathcal{H}$. Here we use the Hausdorff deviation function $\mathrm{dist}_{\mathcal{H}}(A_1, A_2)$ for subsets $A_1, A_2 \subset \mathcal{H}$ defined as follows:

$$\mathrm{dist}_{\mathcal{H}}(A_1, A_2) = \sup_{a_1 \in A_1} \inf_{a_2 \in A_2} \|a_1 - a_2\|_{\mathcal{H}}.$$

A global attractor of the system is a minimal invariant set which attracts all bounded sets of initial data. It describes the system's dynamics when a sufficiently large period of time has passed. Therefore, the global attractor contains the information of all limiting spatio-temporal regimes the system possesses. It may be used for investigation of the "information capacity" for delayed systems.

4. FINITE-DIMENSIONAL PROPERTY OF ATTRACTORS

In the present paragraph, we discuss two characteristics of finite-dimensionality of the attractors, namely the number of determining modes and Hausdorff dimension. The first one is suggested in [17] to describe finite-dimensional dynamics of Navier-Stokes systems, and further developed in [18]. Application of Hausdorff dimension of attractors for distributed parameters systems is a subject of a great number of articles (see [14–16,18] and others). Concerning the retarded systems can only refer to [19], where the finiteness of topological dimension is obtained for any negatively-invariant set of delayed system of ordinary differential equations.

Here we use methods of [12] and [19] and combine them with a technique developed in [17] and [18]. As a result, we obtain estimates for the number of determining modes and Hausdorff dimension of attractors in a retarded system of differential-difference equations (3)–(5) and generalize them to the attractors of retarded diffusion equations (7)–(10).

Starting up with the notion of determining modes, let us consider attractor $\mathcal{U}$ for definiteness. Note that element $\mathbf{\Phi} = (\mathbf{\Phi}^{(-1)}, \mathbf{\Phi}^{(0)})$ arbitrarily given in $\mathcal{H}$ possesses the expansion into series

$$\mathbf{\Phi} = \left(\sum_{j=0}^{+\infty} \mathbf{p}_j c_j(t), 0 \right) + \left(0, \mathbf{\Phi}^{(0)}\right),$$

where

$$\mathbf{p}_j = \int_0^T \mathbf{\Phi}^{(-1)}(\tau)\, c_j(\tau)\, \mathrm{d}\tau, \quad c_0(\tau) = \sqrt{\frac{1}{T}}, \quad c_j(\tau) = \sqrt{\frac{2}{T}} \cos\left(\frac{\pi j \tau}{T}\right), \qquad j = 1, \dots .$$

We introduce the projection operators $\mathcal{P}_N$ and $\mathcal{P}_N^\perp$ according to the formulas

$$\mathcal{P}_N\mathbf{\Phi} = \left(\sum_{j=0}^{N} \mathbf{p}_j c_j(t), 0\right) + \left(0, \mathbf{\Phi}^{(0)}\right),$$
$$\mathcal{P}_N^\perp\mathbf{\Phi} = \left(\sum_{j=N+1}^{+\infty} \mathbf{p}_j c_j(t), 0\right).$$

One can see that $\mathcal{P}_N + \mathcal{P}_N^\perp$ is identical operator in $\mathcal{H}$, and linear dimension $\dim \mathcal{P}_N\mathcal{H}$ is equal to $(n+1)N$.

Following [17,18] we say that attractor $\mathcal{U}$ possesses a finite number of determining modes, if there exists an integer number N such that every pair of orbits

$$\{\mathbf{\Phi}_k\} = \{\mathbf{\Phi}_k : \mathbf{\Phi}_k = S\mathbf{\Phi}_{k-1}\}_{k=-\infty}^{+\infty} \in \mathcal{U}, \qquad \{\mathbf{\Psi}_k\} = \{\mathbf{\Psi}_k : \mathbf{\Psi}_k = S\mathbf{\Psi}_{k-1}\}_{k=-\infty}^{+\infty} \in \mathcal{U},$$

satisfying the property

$$\mathcal{P}_N\left(\mathbf{\Phi}_k - \mathbf{\Psi}_k\right) = 0, \qquad k = 0, \pm 1, \pm 2, \dots, \tag{19}$$

must coincide, i.e., $\mathbf{\Phi}_k - \mathbf{\Psi}_k = 0$, $k = 0, \pm 1, \pm 2, \dots$. In other words, if the attractor possesses a finite number of determining modes, then every orbit lying on the attractor may be reconstructed by finite number of its coefficients of determining modes. From the physical standpoint, this means that "lower" spatio-temporal modes contain almost all information of the system's dynamics on the global attractor, and in some sense "higher" modes play a slaving role (a kind of slaving principle). Knowing the number of determining modes while computer modelling, one must choose parameters of the numerical algorithm correctly: a digital mesh must resolve all the determining scales.

THEOREM 1. *The upper estimate of the number of determining modes of the attractor $\mathcal{U}$, is given by the formula*

$$N \le N_{\mathrm{det}}(\mathcal{U}) \equiv \left[2\pi^{-1} \max\{1, nM_1\}T\right] + 1.$$

PROOF. Our conclusions are based on the following inequality, obtained in the Appendix:

$$\left\|\mathcal{P}_N^\perp(S\mathbf{\Phi} - S\mathbf{\Psi})\right\|_{\mathcal{H}} \le \delta\|\mathbf{\Phi} - \mathbf{\Psi}\|_{\mathcal{H}}, \qquad \mathbf{\Phi}, \mathbf{\Psi} \in \mathcal{U}, \tag{20}$$

where

$$\delta = \frac{2\max\{1, nM_1\}T}{\pi N}.$$

Fixing $N = N_{\mathrm{det}}(\mathcal{U})$ we get the number N to produce the condition $0 < \delta < 1$ of "squeezing" property.

Thus, if two orbits $\{\mathbf{\Phi}_k\}$ and $\{\mathbf{\Psi}_k\}$ lie on $\mathcal{U}$ and satisfy (19), then $\mathbf{\Phi} - \mathbf{\Psi} = \mathcal{P}_N^\perp(\mathbf{\Phi} - \mathbf{\Psi})$. Arguing (20) and the relationships $\mathbf{\Phi}_k = S^m\mathbf{\Phi}_{k-m}$ and $\mathbf{\Psi}_k = S^m\mathbf{\Psi}_{k-m}$, we obtain

$$\left\|\mathcal{P}_N^\perp\left(\mathbf{\Phi}_k - \mathbf{\Psi}_k\right)\right\|_{\mathcal{H}} \le \delta^m \left\|\mathbf{\Phi}_{k-m} - \mathbf{\Psi}_{k-m}\right\|_{\mathcal{H}}, \qquad m = 1, 2, \dots.$$

Elements $\mathbf{\Phi}_{k-m}$ and $\mathbf{\Psi}_{k-m}$ are bounded in $\mathcal{H}$ as corresponding parts of orbits $\{\mathbf{\Phi}_k\}$ and $\{\mathbf{\Psi}_k\}$ lying on the attractor $\mathcal{U}$. In particular, they belong to the ball in $\mathcal{H}$ centered the in origin with radius $\rho_0 = (1 + nM_0^2(1+T))^{1/2}$. Then

$$\left\|\mathcal{P}_N^\perp\left(\mathbf{\Phi}_k - \mathbf{\Psi}_k\right)\right\|_{\mathcal{H}} \le 2\rho_0\delta^m, \qquad m = 1, 2, \dots. \tag{21}$$

As $0 < \delta < 1$ and m tends to $+\infty$, the right part of (21) tends to zero, and the left one does the same. Finally, taking into account the condition (19), we get the equalities $\mathbf{\Phi}_k - \mathbf{\Psi}_k = 0$, $k = 0, \pm 1, \pm 2, \dots$ as desired. This ends of proof.

COROLLARY. *One can see that our estimate for the number of determining mode increases linearly with time delay. This fact is in a good accordance with the estimates of Lyapunov dimension of chaotic attractor mentioned in [10].*

Observing the proof of Theorem 1 lets us emphasize the inequality (20). It describes the "squeezing" property of operator S on the attractor $\mathcal{U}$. Precisely, operator S produces the squeezing of dimensions (scales) along almost all directions in the space $\mathcal{H}$ except the finite number of the attractor's determining modes.

We have also generalized this approach to the delayed diffusion system with spatially-distributed parameters. Some difficulties while examining of the squeezing property are removed in the Appendix by applying the fractional derivatives technique.

To construct projection operators in $\mathcal{H}_D$, we first introduce the following sets of indexes:

$$I_0 = \{j \in \mathcal{Z} : |j| \le N_0\}, \qquad I_{-1} = \{(l,m) \in \mathcal{Z} \times \mathcal{N} : |l| \le N_1,\ m \le N_2\},$$
$$I_0^{\perp} = \{j \in \mathcal{Z} : |j| > N_0\}, \qquad I_{-1}^{\perp} = \{(l,m) \in \mathcal{Z} \times \mathcal{N} : |l| > N_1 \text{ or } m > N_2\}.$$

Projection operators $\mathcal{P}_{N_0,N_1,N_2}$ and $\mathcal{P}^{\perp}_{N_0,N_1,N_2}$ are determined from the equalities

$$\mathcal{P}_{N_0,N_1,N_2}\Phi = \left(\sum_{(l,m)\in I_{-1}} p_{l,m}^{(-1)} e_l(x) c_m(t),\ \sum_{j\in I_0} p_j^{(0)} e_j(x) \right),$$
$$\mathcal{P}^{\perp}_{N_0,N_1,N_2}\Phi = \left(\sum_{(l,m)\in I_{-1}^{\perp}} p_{l,m}^{(-1)} e_l(x) c_m(t),\ \sum_{j\in I_0^{\perp}} p_j^{(0)} e_j(x) \right),$$

where $\Phi = (\varphi^{(-1)}, \varphi^{(0)}) \in \mathcal{H}_D$, $e_l(x) = (2\pi)^{-1/2}\exp(ilx)$,

$$p_{l,m}^{(-1)} = \left\langle \varphi^{(-1)},\, e_l^* c_m \right\rangle_{\Omega_T}, \qquad p_j^{(0)} = \left\langle \varphi^{(0)},\, e_j^* \right\rangle_{\Omega}, \tag{22}$$

and superscript "*" denotes a sign of complex-conjugate value.

We say that attractor $\mathcal{U}_D$ of retarded diffusion system (7)–(10) possesses a finite number of determining modes, if there exists a triplet of numbers $(N_0, N_1, N_2) \in \mathcal{Z} \times \mathcal{Z} \times \mathcal{N}$ such that every pair of orbits $\{\Phi_k\}, \{\Psi_k\} \in \mathcal{U}_D$, satisfying the property

$$\mathcal{P}_{N_0,N_1,N_2}(\Phi_k - \Psi_k) = 0, \qquad k = 0, \pm 1, \pm 2, \dots,$$

must coincide, i.e., $\Phi_k - \Psi_k = 0$, $k = 0, \pm 1, \pm 2, \dots$.

THEOREM 2. *Let the numbers N_0, N_1, N_2 be chosen from the conditions $\delta_{-1} < 1$, $\delta_0 < 1$, where*

$$\delta_{-1}^2 = M_1^2 \left(\frac{1}{2DN_0^2}\left(1 + \frac{\min\{1/4, T\}}{T^2}\right) + \frac{1}{4DN_1^2} + \frac{3T}{\pi N_2} \right),$$
$$\delta_0^2 = \frac{\min\{1/4, T\}}{N_0^2 D T^2} + \frac{1}{2DN_1^2} + \frac{3T}{\pi N_2}.$$

Then attractor $\mathcal{U}_D$ possesses a finite number of determining modes estimated by the value $N_D = 2N_0 + 1 + (2N_1 + 1)(N_2 + 1)$.

PROOF. Basing on the inequality

$$\left\|\mathcal{P}^{\perp}_{N_0,N_1,N_2}(S\Phi - S\Psi)\right\|^2_{\mathcal{H}_D} \le \delta_{-1}^2 \left\|\varphi^{(-1)} - \psi^{(-1)}\right\|^2_{\Omega_T} + \delta_0^2 \left\|\varphi^{(0)} - \psi^{(0)}\right\|^2_{\Omega}, \tag{23}$$

obtained in the Appendix, the proof may be fulfilled similarly to the one from Theorem 1.

Another characteristic of finite-dimensional dynamics is connected with Hausdorff dimension of attractors. Let us remind this notion taking for definiteness the compact set $\mathcal{U} \in \mathcal{H}$. Let $\mathcal{U}_\epsilon = \{B_j\}$ be an arbitrary cover of $\mathcal{U}$ by means of the balls $B_j = \{\mathbf{\Phi} \in \mathcal{H} : \|\mathbf{\Phi} - \mathbf{\Phi}_j\|_{\mathcal{H}} \leq \rho_j\}$ of radii $\rho_j \leq \epsilon$, centered in $\mathbf{\Phi}_j \in \mathcal{H}$. The value

$$h_M(\mathcal{U}) = \lim_{\epsilon \to 0} \inf_{\mathcal{U}_\epsilon} \sum_j \rho_j^M$$

is Hausdorff dimension measure M of the set $\mathcal{U}$. The minimum of those M, for which $h_M(\mathcal{U})$ is finite, is referred to as Hausdorff dimension of $\mathcal{U}$:

$$\dim_H \mathcal{U} = \inf \{M : h_M(\mathcal{U}) < +\infty\}.$$

Direct applying of this formula to the attractor $\mathcal{U}$ seems to be quite difficult. But the squeezing property provides the possibility to get the upper estimate of the Hausdorff dimension according the following statement.

LEMMA. *(See [18].) Let $\mathcal{A}$ be a bounded subset in Hilbert space $\mathcal{H}$, $\mathcal{A} \subseteq S(\mathcal{A})$, where S is an operator acting on $\mathcal{A}$. If there exists a finite-dimensional projection operator $\mathcal{P}$, $\dim \mathcal{P}\mathcal{H} = m$ and for every $a, \hat{a} \in \mathcal{A}$ the inequalities*

$$\|Sa - S\hat{a}\|_{\mathcal{H}} \leq \ell \, \|a - \hat{a}\|_{\mathcal{H}}, \qquad \ell > 0, \tag{24}$$

$$\left\|\mathcal{P}^{\perp} (Sa - S\hat{a})\right\|_{\mathcal{H}} \leq \delta \, \|a - \hat{a}\|_{\mathcal{H}}, \qquad 0 < \delta < 1, \tag{25}$$

hold, then $\dim_H \mathcal{A} \leq d(m, \ell, \delta)$, where

$$d(m, \ell, \delta) = m \ln \left(\frac{8\chi^2 \ell^2}{1 - \delta^2}\right) \ln^{-1} \left(\frac{2}{1 + \delta^2}\right), \tag{26}$$

and χ is Gauss constant.

As a result, we formulate the following theorem.

THEOREM 3. *Attractors $\mathcal{U}$ and $\mathcal{U}_D$ possess a finite value of Hausdorff dimension estimated according to the inequalities*

$$\dim_H \mathcal{U} \leq d\left((n+1)N, \max\{1, nM_1\}, \delta\right),$$

$$\dim_H \mathcal{U}_D \leq d\left(N_D, \max\{1, M_1\}, \max\left(\delta_{-1}, \delta_0\right)\right),$$

where N, δ and N_D, δ_{-1}, δ_0 are given from Theorems 1 and 2.

PROOF. Let us apply the lemma where attractors $\mathcal{U}$ and $\mathcal{U}_D$ are given as subset $\mathcal{A}$ of corresponding spaces $\mathcal{H}$ and $\mathcal{H}_D$. Examining the conditions (25), we note that (26) is provided by applying the projection operators $\mathcal{P}_N$, $\mathcal{P}_{N_0,N_1,N_2}$ and squeezing estimates (20) and (23). Then, referring to the Appendix to obtain the inequalities (24) with values $\ell = \max\{1, nM_1\}$ for operator S, and $\ell = \max\{1, M_1\}$ for operator S_D we use (26) and complete the proof.

5. CONCLUSION

A mathematical approach for evaluating the complexity of spatio-temporal dynamics for delayed feedback optical systems is suggested. It is based on the understanding of the fact that a global attractor describes the whole of the system's dynamics after a large period of time. The universal technique for proving attractor's existence and estimating the number of determining modes and Hausdorff dimension is developed for delayed optical systems. This technique is based on the reducing of a retarded system to discrete dissipative semigroup of shifting operators and on obtaining the squeezing property. Thus, both the retarded differential-difference and retarded diffusion models of feedback optical systems are similarly investigated. As a result, new upper estimates for the number of determining modes and Hausdorff dimension of global attractors are obtained as functions of nonlinearity, delay value, and diffusion. These estimates are applicable in evaluating the complexity and information capacity of delayed systems. Another useful application is connected with choosing the dimension of discrete numerical mesh while computer simulation of optical systems: a mesh must resolve all the determining modes.

APPENDIX

A.1. Obtaining Inequality (17)

Taking scalar product (11) with $\mathbf{u}(t)$ in R^n after some algebra we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{u}(t)\right\|_n^2 + \left\|\mathbf{u}(t)\right\|_n^2 \le \left\|\mathbf{F}\left(\mathbf{\Phi}^{(-1)}(t)\right)\right\|_n^2 \le nM_0^2. \tag{27}$$

Multiplying (27) by $\exp(t)$, integrating the result over $(0,t)$ and using (12), we get

$$\left\|\mathbf{u}(t)\right\|_n^2 \le \left\|\mathbf{\Phi}^{(0)}\right\|_n^2 \exp(-t) + nM_0^2\left(1-\exp(-t)\right). \tag{28}$$

In particular, when $t = T$ the following inequality holds:

$$\left\|\mathbf{u}(T)\right\|_n^2 \le \left\|\mathbf{\Phi}^{(0)}\right\|_n^2 \exp(-T) + nM_0^2\left(1-\exp(-T)\right). \tag{29}$$

Integrating (28) over $(0,T)$, we obtain

$$\left\|\mathbf{u}\right\|_{(0,T),n}^2 \le \left\|\mathbf{\Phi}^{(0)}\right\|_n^2 \left(1-\exp(-T)\right) + nM_0^2\left(T-1+\exp(-T)\right). \tag{30}$$

Recalling that $S\mathbf{\Phi} = (\mathbf{u}, \mathbf{u}(T))$, $S^k = S(S^{k-1})$ and arguing (29)–(30), we get the following inequalities by induction on k:

$$\left\|S^k\mathbf{\Phi}\big|_{(0)}\right\|_n^2 \le \left\|\mathbf{\Phi}^{(0)}\right\|_n^2 \exp(-kT) + nM_0^2\left(1-\exp(-kT)\right), \tag{31}$$

$$\begin{aligned}\left\|S^k\mathbf{\Phi}\big|_{(-1)}\right\|_{(0,T),n}^2 \le{}& \left\|\mathbf{\Phi}^{(0)}\right\|_n^2 \left(\exp\left(-(k-1)T\right)-\exp(-kT)\right)\\ &+ nM_0^2\left(T-\exp\left(-(k-1)T\right)+\exp(-kT)\right).\end{aligned} \tag{32}$$

Taking into account that $S^k\mathbf{\Phi} = (S^k\mathbf{\Phi}|_{(-1)}, S^k\mathbf{\Phi}|_{(0)})$ and summing (31),(32), we obtain (17).

A.2. Obtaining Inequality (18)

Our method is similar to the one applied in Section 5.1. Multiplying (14) by $u(t)$ in $L^2(\Omega)$ and using (10) after some algebra, we get

$$\frac{\partial}{\partial t}\left\|u(t)\right\|_\Omega^2 + \left\|u(t)\right\|_\Omega^2 + 2D\left\|u_x(t)\right\|_\Omega^2 \le 2\pi M_0^2, \tag{33}$$

where u_x denotes the partial derivative in respect to x. Multiplying (33) by $\exp(t)$, integrating the result over $(0,T)$ and using (15) we obtain the spatially-distributed analog of (30):

$$\left\|u(t)\right\|_\Omega^2 \le \left\|\varphi^{(0)}\right\|_\Omega^2 \exp(-t) + 2\pi M_0^2\left(1-\exp(-t)\right).$$

Further conclusions are similar to (29)–(32).

A.3. Examining Estimates (20) and (24) for Operator S

If $\mathbf{\Psi} = (\mathbf{\Psi}^{(-1)}, \mathbf{\Psi}^{(0)})$ then vector-valued function $\mathbf{v}(t) = \mathbf{u}(t;\mathbf{\Phi}) - \mathbf{u}(t;\mathbf{\Psi})$ satisfies the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) + \mathbf{v}(t) = \partial\mathbf{F}\left(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{z}^*\right)\left(\mathbf{\Phi}^{(-1)}(t) - \mathbf{\Psi}^{(-1)}(t)\right), \qquad 0 < t < T, \tag{34}$$

$$\mathbf{v}(0) = \mathbf{\Phi}^{(0)} - \mathbf{\Psi}^{(0)}, \tag{35}$$

where $\partial\mathbf{F}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{z}^*)$ denotes the matrix of partial derivatives $\{\frac{\partial f_i(\mathbf{z}_i^*)}{\partial z_j},\ i,j=1,\ldots,n\}$, $\mathbf{z}_i^*=\theta_i\mathbf{\Phi}^{(-1)}+(1-\theta_i)\mathbf{\Psi}^{(-1)}$ and $\theta_i\in(0,1)$ are given by the Lagrange formula. Then, multiplying (34) in R^n by $\mathbf{v}(t)$ and using (4),(35) we get the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|_n^2+\|\mathbf{v}(t)\|_n^2\le n^2M_1^2\left\|\mathbf{\Phi}^{(-1)}(t)-\mathbf{\Psi}^{(-1)}(t)\right\|_n^2.$$

After integrating over $(0,T)$, we obtain

$$\|\mathbf{v}(T)\|_n^2+\|\mathbf{v}\|_{(0,T),n}^2\le\left\|\mathbf{\Phi}^{(0)}-\mathbf{\Psi}^{(0)}\right\|_n^2+n^2M_1^2\left\|\mathbf{\Phi}^{(-1)}-\mathbf{\Psi}^{(-1)}\right\|_{(0,T),n}^2. \tag{36}$$

In terms of $\mathcal{H}$-norm, (36) takes the form

$$\|S\mathbf{\Phi}-S\mathbf{\Psi}\|_{\mathcal{H}}\le\max\{1,nM_1\}\|\mathbf{\Phi}-\mathbf{\Psi}\|_{\mathcal{H}}.$$

This inequality is valid for every $\mathbf{\Phi},\mathbf{\Psi}\in\mathcal{H}$, and, in particular, for $\mathbf{\Phi}$ and $\mathbf{\Psi}$ lying on the attractor $\mathcal{U}$. Thus, (24) holds.

Starting the examining of the squeezing property (20) we represent $\mathcal{P}_N^{\perp}(S\mathbf{\Phi}-S\mathbf{\Psi})$ in the form of series expansion

$$\mathcal{P}_N^{\perp}(S\mathbf{\Phi}-S\mathbf{\Psi})=\left(\sum_{j=N+1}^{+\infty}\mathbf{v}_jc_j(t),0\right),$$

where $\mathbf{v}_j=(v_{j,1},v_{j,2},\ldots,v_{j,n})\in R^n$ are Fourier coefficients of $\mathbf{v}(t)$, $t\in(0,T)$ with respect to the system $\{c_j(t)\}$ orthogonal in $L_2(0,T)$. As the series expansion converges uniformly on $[0,T]$ and there exists the derivative $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\in L_2^n(0,T)$, then the following formula holds:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}(t)=-\frac{\pi}{T}\sum_{j=1}^{+\infty}j\,\mathbf{v}_j\,s_j(t),\qquad s_j(t)=\sqrt{\frac{2}{T}}\sin\frac{\pi j}{T}t.$$

Substituting it into (34), we get

$$-\frac{\pi}{T}\sum_{j=1}^{+\infty}j\,\mathbf{v}_j\,s_j(t)+\mathbf{v}(t)=\partial\mathbf{F}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{z}^*)\left(\mathbf{\Phi}^{(-1)}(t)-\mathbf{\Psi}^{(-1)}(t)\right). \tag{37}$$

Multiplying (37) by $\mathbf{v}_l\,s_l(t)$ in $L_2^n(0,T)$ and arguing that s_l is orthogonal to s_j, $j\neq l$, we obtain

$$\frac{\pi}{T}l\,\|\mathbf{v}_l\|_n^2=\int_0^T\left\langle\mathbf{v}(t)-\partial\mathbf{F}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{z}^*)\left(\mathbf{\Phi}^{(-1)}(t)-\mathbf{\Psi}^{(-1)}(t)\right),\mathbf{v}_ls_l(t)\right\rangle_n\mathrm{d}t.$$

Summing these equations over $l=N+1,N+2,\ldots$ and using (4) we get

$$\begin{aligned}\frac{\pi}{T}N\sum_{l=N+1}^{+\infty}\|\mathbf{v}_l\|_n^2&\le\frac{\pi}{T}\sum_{l=N+1}^{+\infty}l\,\|\mathbf{v}_l\|_n^2\\&\le\left(\|\mathbf{v}\|_{(0,T),n}+nM_1\left\|\mathbf{\Phi}^{(-1)}-\mathbf{\Psi}^{(-1)}\right\|_{(0,T),n}\right)\left(\sum_{l=N+1}^{+\infty}\|\mathbf{v}_l\|_n^2\right)^{1/2}.\end{aligned}$$

In order to estimate $\|\mathbf{v}\|_{(0,T),n}$ we argue to (36) and deduce

$$\left(\frac{\pi}{T}N\right)^2\sum_{l=N+1}^{+\infty}\|\mathbf{v}_l\|_n^2\le2\left\|\mathbf{\Phi}^{(0)}-\mathbf{\Psi}^{(0)}\right\|_n^2+4n^2M_1^2\left\|\mathbf{\Phi}^{(-1)}-\mathbf{\Psi}^{(-1)}\right\|_{(0,T),n}^2.$$

Taking into account that the left part of the inequality is equal to $(\pi NT^{-1})^2\|\mathcal{P}_N^{\perp}(S\mathbf{\Phi}-S\mathbf{\Psi})\|_{\mathcal{H}}^2$, after some algebra we come to (20).

A.4. Examining Estimates (23) and (24) for Operator S_D

Let $\Phi = (\varphi^{(-1)}, \varphi^{(0)}) \in \mathcal{H}_D$ and $\Psi = (\psi^{(-1)}, \psi^{(0)}) \in \mathcal{H}_D$. Then $S_D\Phi - S_D\Psi = (v, v(T)) \in \mathcal{H}_D$, where $v = v(x,t)$ is 2π-periodical in x solution of the system

$$\frac{\partial v}{\partial t} + v = D\frac{\partial^2 v}{\partial x^2} + \partial F(\Phi, \Psi, \Delta, \xi)\left(\varphi^{(-1)}(x+\Delta, t) - \psi^{(-1)}(x+\Delta, t)\right), \tag{38}$$

$$v(x,0) = \varphi^{(0)}(x) - \psi^{(0)}(x), \tag{39}$$

where $\partial F(\Phi, \Psi, \Delta, \xi) = F'(\xi\varphi^{(-1)}(x+\Delta,t) + (1-\xi)\psi^{(-1)}(x+\Delta,t))$ and ξ is obtained from the Lagrange formula. Multiplying (38) by $v(t)$, integrating the result over Ω_T and using (39) we come to the inequality

$$\|v(T)\|_\Omega^2 + 2\|v\|_{\Omega_T}^2 + 2D\|v_x\|_{\Omega_T}^2 \le \left\|\varphi^{(0)} - \psi^{(0)}\right\|_\Omega^2 + 2M_1\|v\|_{\Omega_T}\left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}. \tag{40}$$

Estimating the second term with the sum $M_1^2\|\varphi^{(-1)} - \psi^{(-1)}\|_{\Omega_T}^2 + \|v\|_{\Omega_T}^2$, we have

$$\|v(T)\|_\Omega^2 + \|v\|_{\Omega_T}^2 \le M_1^2\left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2 + \left\|\varphi^{(0)} - \psi^{(0)}\right\|_\Omega^2. \tag{41}$$

Then the inequality

$$\|S_D\Phi - S_D\Psi\|_{\mathcal{H}_D} \le \max\{1, M_1\}\|\Phi - \Psi\|_{\mathcal{H}_D}$$

holds, and the condition (24) is satisfied.

To derive squeezing property of S_D on the attractor $\mathcal{U}_D$, let us consider projection operators Q_{N_0} and Q_{N_1,N_2}, defined according to the formulas

$$Q_{N_0}b = \sum_{j\in I_0^\perp} b_j\, e_j(x), \qquad b \in L^2(\Omega),$$

$$Q_{N_1,N_2}h = \sum_{(l,m)\in I_{-1}^\perp} h_{l,m}\, e_l(x)\, c_m(t), \qquad h \in L^2(\Omega_T).$$

Here, b_j and $h_{l,m}$ are Fourier coefficients of $b(x)$ and $h(x,t)$ (see (22)). Then the following equality holds:

$$\left\|\mathcal{P}_{N_0,N_1,N_2}^\perp(S_D\Phi - S_D\Psi)\right\|_{\mathcal{H}_D}^2 = \|Q_{N_0}v(T)\|_\Omega^2 + \|Q_{N_1,N_2}v\|_{\Omega_T}^2. \tag{42}$$

Let us estimate the first term in the sum (42). To this end we use the inequality

$$\|v_x\|_{\Omega_T}^2 \le \frac{M_1^2}{4D}\left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2 + \frac{1}{2D}\left\|\varphi^{(0)} - \psi^{(0)}\right\|_\Omega^2. \tag{43}$$

It follows from (40) and from the inequality

$$2M_1\|v\|_{\Omega_T}\left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T} \le 0.5M_1^2\left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2 + 2\|v\|_{\Omega_T}^2.$$

Then, multiplying (38) by $-t^2\frac{\partial^2 v}{\partial x^2}$ in $L^2(\Omega)$ and using (10) we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\left(t^2\,\|v_x(t)\|_\Omega^2\right) \le \left(t - t^2\right)\|v_x(t)\|_\Omega^2 + \frac{t^2M_1^2}{2D}\left\|\varphi^{(-1)}(t) - \psi^{(-1)}(t)\right\|_\Omega^2.$$

After integrating over $(0,T)$ and using (43) and the estimate $t - t^2 \le \min\{1/4, T\}$, $t \in (0,T)$ we get

$$\|v_x(T)\|_\Omega^2 \le \frac{\min\{1/4,T\}}{DT^2}\left\|\varphi^{(0)} - \psi^{(0)}\right\|_\Omega^2 + \frac{M_1^2}{2D}\left(\frac{\min\{1/4,T\}}{T^2} + 1\right)\left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2.$$

Consider an expansion of $v(x,T)$ into series $v(x,T) = \sum_{k=-\infty}^{+\infty} b_k\, e_k(x)$. As $v_x(x,T) \in L^2(\Omega)$, the equality $v_x(x,T) = \sum_{l=-\infty}^{+\infty} i\, l\, b_l\, e_l(x)$ is valid. Therefore,

$$\|Q_{N_0} v(T)\|_\Omega^2 = \sum_{l\in I_0^\perp} |b_l|^2 \le N_0^{-2} \sum_{l\in I_0^\perp} l^2\, |b_l|^2 \le N_0^{-2}\, \|v_x(T)\|_\Omega^2\,,$$

and thus, the following inequality holds:

$$\begin{aligned} \|Q_{N_0} v(T)\|_\Omega^2 \le \frac{\min\{1/4, T\}}{N_0^2 D T^2} \left\|\varphi^{(0)} - \psi^{(0)}\right\|_\Omega^2 \\ + \frac{M_1^2}{2DN_0^2} \left(\frac{\min\{1/4,T\}}{T^2} + 1\right) \left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2 . \end{aligned} \tag{44}$$

To estimate the second term in (42), let us represent it in the form of series expansion

$$\|Q_{N_1,N_2} v\|_{\Omega_T}^2 = \sum_{|l|>N_1} \sum_{m=0}^{+\infty} |v_{l,m}|^2 + \sum_{|l|\le N_1} \sum_{m=N_2+1}^{+\infty} |v_{l,m}|^2 = \sum\nolimits_1 + \sum\nolimits_2 .$$

Note that

$$\sum\nolimits_1 \le \frac{1}{N_1^2} \sum_{l=-\infty}^{+\infty} \sum_{m=0}^{+\infty} l^2\, |v_{l,m}|^2 = \frac{1}{N_1^2}\, \|v_x\|_{\Omega_T}^2 .$$

Then, using (43) we obtain the estimate

$$\sum\nolimits_1 \le \frac{M_1^2}{4DN_1^2} \left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2 + \frac{1}{2DN_1^2} \left\|\varphi^{(0)} - \psi^{(0)}\right\|_\Omega^2 .$$

To get an estimate of $\sum_2$ we consider scalar valued function $v_l(t) = \langle v(\cdot,t), e_l\rangle_\Omega$. As there exists the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} v_l(t) = \left\langle \frac{\partial v}{\partial t}(\cdot,t), e_l \right\rangle_\Omega ,$$

then the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, v_l(t) = -\left(1 + D\, l^2\right) v_l(t) + \langle \mathcal{F}(\cdot,t), e_l\rangle_\Omega . \tag{45}$$

Here we denote $\mathcal{F} = \partial F(\Phi,\Psi,\Delta,\xi)\,(\varphi^{(-1)}(x+\Delta,t) - \psi^{(-1)}(x+\Delta,t))$. It may be derived from (38) after its multiplying by $e_l(x)$ and integrating over Ω. Basing on (45) and on expansions

$$v_l(t) = \sum_{m=0}^{+\infty} v_{l,m}\, c_m(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t} v_l(t) = -\sum_{m=1}^{+\infty} \frac{\pi m}{T} v_{l,m}\, s_m(t),$$

we obtain the equalities

$$\frac{\pi m}{T}\, v_{l,m} = \left(1 + D\, l^2\right) \hat{v}_{l,m} - \hat{\mathcal{F}}_{l,m}, \tag{46}$$

where $\hat{f}_{l,m} = (\pi T)^{-1} \langle f, e_l^* s_m\rangle_{\Omega_T}$ denotes the corresponding Fourier coefficient of $f \in \mathcal{H}_D$ with respect to the system $\{e_l(x)\, s_m(t)\}$. Multiplying (46) by $v_{l,m}^*$ after some algebra we get

$$m\, |v_{l,m}|^2 \le \frac{T}{\pi} \left(\left(1 + \frac{1}{2} D\, l^2\right) \left(|\hat{v}_{l,m}|^2 + |v_{l,m}|^2\right) + \frac{1}{3} \left|\hat{\mathcal{F}}_{l,m}\right|^2 \right).$$

Summing over $|l| \le N_1$, $m > N_2$ we obtain

$$\sum\nolimits_2 \le \frac{T}{N_2 \pi} \left(2\|v\|_{\Omega_T}^2 + D\, \|v_x\|_{\Omega_T}^2 + \frac{M_1^2}{3} \left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2 \right).$$

To complete our conclusions we use (41) and (43) and come to the inequality

$$\sum\nolimits_2 \leq \frac{3TM_1^2}{N_2\pi} \left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2 + \frac{3T}{N_2\pi} \left\|\varphi^{(0)} - \psi^{(0)}\right\|_{\Omega}^2.$$

Finally, arguing (44) and the estimates of $\sum_{1,2}$, we get the inequality of squeezing

$$\begin{aligned}
&\left\|\mathcal{P}^{\perp}_{N_0,N_1,N_2}(S_D\Phi - S_D\Psi)\right\|_{\mathcal{H}_D}^2 \\
&\quad \leq M_1^2 \left(\frac{1}{2DN_0^2}\left(1 + \frac{\min\{1/4, T\}}{T^2}\right) + \frac{1}{4DN_1^2} + \frac{3T}{\pi N_2}\right) \left\|\varphi^{(-1)} - \psi^{(-1)}\right\|_{\Omega_T}^2 \\
&\quad + \left(\frac{\min\{1/4, T\}}{N_0^2 D T^2} + \frac{1}{2DN_1^2} + \frac{3T}{\pi N_2}\right) \left\|\varphi^{(0)} - \psi^{(0)}\right\|_{\Omega}^2.
\end{aligned}$$

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